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Note

Integrality properties of edge path tree families

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ABSTRACT

An Edge Path Tree (EPT) family is a family whose members are edge sets of paths in a tree. Relying on the notion of Pie introduced in [M.C. Golumbic, R.E. Jamison, The edge intersection graphs of paths in a tree, Journal of Combinatorial Theory, Series B 38 (1985) 8–22], we characterize Ideal and Mengerian EPT families. In particular, we show that an EPT family is Ideal if and only if it is Mengerian. If, in addition, the EPT family is uniform, then it is Ideal if and only if it is Unimodular. The latter equivalence generalizes the well-known fact that the edge set of a graph is an Ideal clutter if and only if the graph is bipartite.

1. Introduction

An *Edge Path Tree* (*EPT* for shortness) family, is a family $\mathcal{P}$ whose members are edge sets of paths in a tree. Any such tree[1] will be referred to as a *supporting tree* of $\mathcal{P}$. EPT families are characterized in [3] and the related recognition problem is the well-known Graph Realization problem [1,2,4] (see also Chapter 20 in [6]).

With any family $\mathcal{P}$ of subsets of a common ground set E and $w \in \mathbb{Z}_+^E$ we can associate the following pair of dual linear programs:

$$\min\left\{\sum_{e\in E} w(e)y(e) \mid \sum_{e\in P} y(e) \geq 1\ \forall P \in \mathcal{P},\ y \in \mathbb{R}_+^E\right\} \tag{1}$$

$$\max\left\{\sum_{P\in\mathcal{P}} x(P) \mid \sum_{P\ni e} x(P) \leq w(e)\ \forall e \in E,\ x \in \mathbb{R}_+^{\mathcal{P}}\right\}. \tag{2}$$

The main aim of this paper is to characterize those EPT families $\mathcal{P}$ for which:

(i) (1) has an integral optimal solution for any $w \in \mathbb{Z}_+^E$; hence the polyhedron of (1) is integral;
(ii) (2) has an integral optimal solution for any $w \in \mathbb{Z}_+^E$ i.e., the defining system of (1) is Totally Dual Integral;

In cases (i) and (ii) $\mathcal{P}$ is, respectively, Ideal and Mengerian. Our results rely on the notion of pie introduced by Golumbic and Jamison in [5] in the context of EPT graphs. We base the characterizations of Ideal and Mengerian EPT families on Lovász's 2-matching characterization of Mengerianity (see Theorem 1) and the additional observation (see Lemma 2) that if an EPT family $\mathcal{P}$ does not contain any odd pie as minor then the members of certain 2-matchings in $\mathcal{P}$ can be chosen "as disjoint as possible".

Let us give now some notation used throughout the rest of the paper. For a graph (V, E) we denote by $E(v)$ the set of edges incident to $v \in V$. The difference and the symmetric difference between two sets A and B will be denoted by $A - B$

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[1] The supporting tree of an EPT family need not be unique: the EPT family $\{\{1, 2\}, \{2, 3\}\}$ is supported by a path of length three or by claw.

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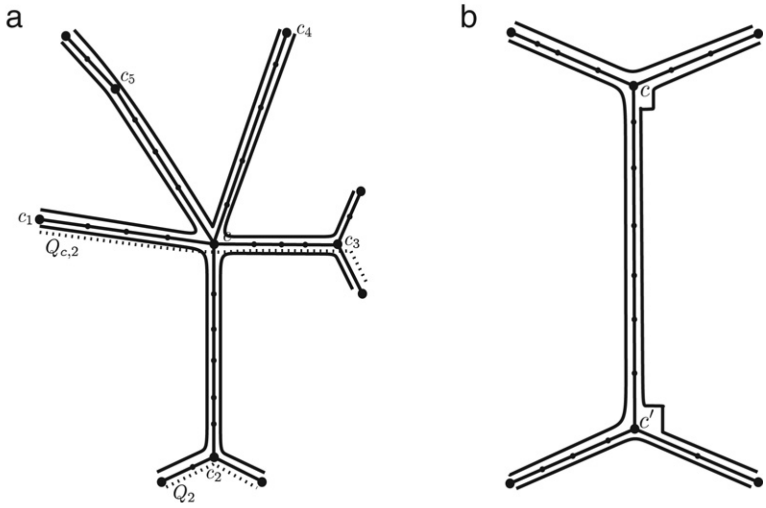

Fig. 1. (a) A pie of size five (thick lines); vertices $c_1, \ldots, c_5$ and paths $Q_{c,2}$ and Q_2 (dotted lines) are defined in Lemma 2; (b) an odd-M-pie-free family which is not odd-pie-free.

and $A \triangle B$, respectively. Let $\mathcal{P}$ be a finite family of subsets of some ground set; $\mathcal{P}$ is a *clutter* if its members are pairwise incomparable w.r.t. set inclusion. Let $E = \cup(P \mid P \in \mathcal{P})$. For $A, B \subseteq E$ and $A \cap B = \emptyset$ the family of the (inclusionwise) minimal members in $\{P - B \mid P \cap A = \emptyset, P \in \mathcal{P}\}$ is denoted by $\mathcal{P} \setminus A/B$ and is referred to as a *minor* of $\mathcal{P}$. If $\mathcal{P}$ is a clutter so is $\mathcal{P} \setminus A/B$. It is well known that $\mathcal{P} \setminus A/B = \mathcal{P}/B \setminus A$. When $A = \emptyset$ or $B = \emptyset$ the notation will be abridged to $\mathcal{P} \setminus A$ (*deletion minor*) and $\mathcal{P}/B$ (*contraction minor*), respectively. With a slight abuse of notation we identify edge sets of paths in a tree with the paths they span. We remark that EPT families are closed under taking subfamilies and minors. Thus, for instance, if $\mathcal{P}$ is an EPT family supported by T and $\mathcal{P}' = \mathcal{P} \setminus A/B$, then $\mathcal{P}'$ is an EPT family supported by a *minor* of T, namely, the tree $T/A \cup B$ obtained by contracting the edges in $A \cup B$.

Let A be a $\{0, 1\}$ matrix; A is *Totally Unimodular* if each of its square submatrices has determinant -1, 0, or $+1$; A is *Balanced* if it does not contain as submatrix the vertex edge incidence matrix of an odd polygon; A is *Totally Balanced* if it does not contain as submatrix the vertex edge incidence matrix of any polygon. Let A be the incidence matrix of a family $\mathcal{P}$ (i.e., the matrix whose columns are the incidence vectors over E of the members of $\mathcal{P}$); $\mathcal{P}$ is *Balanced*, *Unimodular* and *Totally Balanced* if A is Balanced, Totally Unimodular and Totally Balanced, respectively. Let **I**, **M**, **B**, **U** and **TB** denote, respectively, the classes of families that are Ideal, Mengerian, Balanced, Unimodular and Totally Balanced. In general one has $\mathbf{I} \supseteq \mathbf{M} \supseteq \mathbf{B} \supseteq \mathbf{U}$ and $\mathbf{B} \supseteq \mathbf{TB}$ while **U** and **TB** are incomparable. In this paper we show that for EPT families one has $\mathbf{I} = \mathbf{M} \supseteq \mathbf{B} = \mathbf{U} \supseteq \mathbf{TB}$ and for uniform EPT families one has $\mathbf{I} = \mathbf{U}$.

2. Characterizations

A collection $\{P_1, \ldots, P_k\}$ of paths in a tree $T = (V, E)$ is a *pie in T* if, for some $c \in V$ and some set of edges $\{e_1, \ldots, e_k\} \subseteq E(c)$, one has $P_i \cap \{e_1, \ldots, e_k\} = \{e_i, e_{i+1}\}$, $i = 1, \ldots, k$, where, here and throughout the rest of the paper, addition over a set of k indices is modulo k. The pie is *odd* if k is odd; c is the *center* of the pie. An EPT family $\mathcal{P}$ supported by a tree T is *odd-pie-free* if $\mathcal{P}$ contains no subfamily which is a pie in T; $\mathcal{P}$ is *odd-M-pie-free* if there is no minor $\mathcal{P} \setminus A/B$ which is a pie in T' where T' supports $\mathcal{P} \setminus A/B$. These definitions do not depend on the particular supporting tree T. Indeed Golumbic and Jamison [5] proved[2] that if $\mathcal{P}'$ is an odd pie in some tree supporting $\mathcal{P}$ then it is an odd pie in every tree supporting $\mathcal{P}$. In particular, if T' supports $\mathcal{P} \setminus A/B$ then, T' can be chosen as a minor of T and, without loss of generality, $T' \cong T/A \cup B$. Notice that an odd-M-pie-free family need not be odd-pie-free. For instance, the EPT clutter in Fig. 1(b) is odd-M-pie-free but contains two pies of size three centered at c and c', respectively. Let us prove now two preliminary lemmas which we need later (the first one is somehow "folklore"). An orientation ϕ of a simple undirected graph $G = (V, E)$ is a mapping $\phi : E \to V^2$ which assigns each edge $uv \in E$ to exactly one among (u, v) and (v, u). The graph $(V, \phi E)$ is denoted by ϕG.

Lemma 1. *Any odd-pie-free EPT family is Unimodular.*

Proof. We claim that if $\mathcal{P}$ is an odd-pie-free EPT family supported by $T = (V, E)$ then there is an orientation ϕ of T such that $\phi P := \{\phi e \mid e \in P\}$ is the arc set of a directed path in ϕT for each $P \in \mathcal{P}$. The claim implies the proof of the lemma as follows: let $T' = \phi T$; for each $P \in \mathcal{P}$ add to T' an arc e_P from the end to the beginning of ϕP and let $F' = \{e_P \mid P \in \mathcal{P}\}$. Thus the incidence matrix of $\mathcal{P}$ is the *network matrix generated by* $(V, F \cup F')$ and (V, F), F being the arc set of T' (see e.g., [7]

[2] Actually they proved the statement for $k \geq 4$. The proof of the case $k = 3$ (not needed for their purposes) is merely a specialization of their arguments.

Vol. A, p. 213). Any such matrix is Totally Unimodular. So it suffices to prove the claim. We may suppose without loss of generality that $|P| \geq 2$, for each $P \in \mathcal{P}$. Let $\mathcal{L}$ be the family of the edge sets of the subpaths of length two extracted from the paths of $\mathcal{P}$. Observe that ϕP is (the arc set of) a directed path in ϕT for each $P \in \mathcal{P}$ if and only if ϕP is a directed path in ϕT for each $P \in \mathcal{L}$. Moreover, if $\mathcal{P}$ is odd-pie-free so is $\mathcal{L}$. For $v \in V$, denote by $\mathcal{L}(v)$ the set of those members of $\mathcal{L}$ contained in $E(v)$; since $\mathcal{L}$ is odd-pie-free so is $\mathcal{L}(v)$. In particular, the graph $(E(v), \mathcal{L}(v))$ is 2-colorable. Notice that if $\mathcal{P}$ is nonempty then $\mathcal{L}(v)$ is nonempty for some non-leaf vertex $v \in V$. Call any such vertex *interesting*. By orienting the edges in a color class inward the interesting vertex v and those in the other color class outward v, we define an orientation ϕ_0 of the tree spanned by $E(v)$ so that $\phi_0 P$ is a directed path in this directed tree, for each $P \in \mathcal{L}(v)$. Thus, if there is exactly one interesting vertex $v \in V$ then $\mathcal{L} = \mathcal{L}(v)$ and we are done. Let us proceed inductively on the number of interesting vertices of T. Suppose that T has more than one interesting vertex and let v be any of them. Let $E(v) = \{u_1 v, \dots, u_k v\}$. For $i = 1, \dots, k$, remove from T all edges of $E(v)$ but $u_i v$ and let T_i be the unique connected component containing v. Let $\mathcal{L}_i$ denote the family of those members of $\mathcal{L}$ contained in the set of edges of T_i. Thus T_i supports $\mathcal{L}_i$ and $\mathcal{L} = \cup_i \mathcal{L}_i \cup \mathcal{L}(v)$. We know that there is an orientation ϕ_0 of the tree T_0 spanned by $E(v)$ such that for $P \in \mathcal{L}(v)$, $\phi_0 P$ is a directed path in this directed tree. By induction, for $i = 1, \dots, k$, there is an orientation ϕ_i of T_i such that $\phi_i P$ is the arc set of a directed path in $\phi_i T_i$ for each $P \in \mathcal{L}_i$. Moreover, possibly by reversing the orientation of all arcs of $\phi_i T_i$, $\phi_i u_i v = \phi_0 u_i v$. Hence, by pasting the orientations $\phi_0, \dots, \phi_k$ one defines an orientation ϕ of T such that ϕP is a directed path in ϕT for each $P \in \mathcal{L}$ and hence for each $P \in \mathcal{P}$. $\square$

Lemma 1 and the observation that Balanced and Totally Balanced EPT families must be odd-pie-free and pie-free, respectively, immediately imply that for EPT families one has: $\mathbf{B} = \mathbf{U} \supseteq \mathbf{TB}$.

Lemma 2. *Let $\mathcal{P}$ be an odd-M-pie-free EPT clutter supported by a tree T. If $\{P_1, \dots, P_k\} \subseteq \mathcal{P}$ is an odd pie in T then $L_0 \cup L_1 \subseteq P_i \Delta P_{i+1}$ for some $i = 1, \dots, k$ and some two disjoint paths $L_0, L_1 \in \mathcal{P}$.*

Proof. Let $\{P_1, \dots, P_k\}$ be a counterexample centered at c and let T_0 be the subtree spanned by E_0 where $E_0 = \cup_i P_i$. By the definition of pie one has $P_i \cap E_0(c) = \{e_i, e_{i+1}\}$, $i = 1, \dots, k$, where $E_0(c) = \{e_1, \dots, e_k\}$. For $i = 1, \dots, k$, c is an endpoint of $P_i \cap P_{i+1}$. Let c_i be the other endpoint and observe that $P_i \Delta P_{i+1}$ factorizes into two disjoint paths $Q_{c,i}$ and Q_i (one of them possibly trivial) containing c and c_i, respectively, $i = 1, \dots, k$ (see Fig. 1(a)). Remark that $Q_{c,i}$ and Q_i might not belong to $\mathcal{P}$. The set of the leaves of T_0 coincides with the set of the endpoints of $P_1, \dots, P_k$. It follows that in T_0 every path from c to one of its leaves is contained in some P_i. Since $\mathcal{P}$ is a clutter, so is the deletion minor $\mathcal{P}_0 := \mathcal{P} \setminus (E - E_0)$. Thus if a member of $\mathcal{P}_0$ contains c it contains c as inner vertex. Not every member of $\mathcal{P}_0$ contains c as inner vertex. For if not, $|P \cap E_0(c)| = 2$ for every $P \in \mathcal{P}_0$. Since $\{P_1, \dots P_k\}$ is an odd pie, the graph $(E_0(c), \mathcal{P}_0/(E_0 - E_0(c)))$ would contain an odd (hamiltonian) cycle. Thus it would contain an odd circuit induced, say, by $\{e_{i_1}, \dots, e_{i_h}\}$, $h \leq k$. But then the minor obtained from $\mathcal{P}_0/(E_0 - E_0(c))$ by deleting $E_0(c) - \{e_{i_1}, \dots, e_{i_h}\}$ would be an odd pie in some minor of T, contradicting that $\mathcal{P}$ is odd-M-pie-free. Since $\mathcal{P}_0$ is a clutter we conclude that there exists some $i \leq k$ and some $L \in \mathcal{P}_0$ such that $L \subseteq Q_i \subseteq P_i \Delta P_{i+1}$ and L contains c_i as inner vertex. Thus $\{P_i, P_{i+1}, L\}$ is a pie centered at c_i. Let T_1 be the subtree spanned by E_1 where $E_1 = P_i \cup P_{i+1} \cup L$. Since $\mathcal{P}$ is a clutter, so is the deletion minor $\mathcal{P}_1 := \mathcal{P} \setminus (E - E_1)$. Observe that $L \Delta P_i \subseteq P_{i+1}$ and $L \Delta P_{i+1} \subseteq P_i$. Thus, reasoning exactly as above, each $P \in \mathcal{P}_1$ either contains c_i as inner vertex or it is contained in $Q_{c,i} \subseteq P_i \Delta P_{i+1}$ and contains c as inner vertex. The latter case cannot occur because P would be disjoint from L contradicting that $\{P_1, \dots, P_k\}$ is a counterexample. Therefore each member $\mathcal{P}_1$ contains c_i as inner vertex. Thus $\mathcal{P}_1/(E_1 - E_1(c))$ is an odd pie in some minor of T, contradicting that $\mathcal{P}$ is odd-M-pie-free. $\square$

We also need the following theorem due to Lovász (see e.g., [7]) that provides a general characterization of Mengerian clutters. Recall that a *w-matching* x of $\mathcal{P}$ is any integral point in the polyhedron of (2). The number $\sum\{x(P) \mid P \in \mathcal{P}\}$ is called the *size* of x and the maximum size of a w-matching of $\mathcal{P}$ is denoted by $\nu_w(\mathcal{P})$.

Theorem 1 (*Lovász*). *A family $\mathcal{P}$ is Mengerian if and only if $\nu_{2w}(\mathcal{P}) = 2\nu_w(\mathcal{P})$ for each $w \in \mathbb{Z}^E_+$.*

We are now in position to characterize Ideal and Mengerian EPT families. Without loss of generality we may suppose that such families are clutters. Recall that idealness is preserved under taking minors.

Theorem 2. *Let $\mathcal{P}$ be an EPT Clutter. The following statements are equivalent;* (i) *$\mathcal{P}$ is ideal;* (ii) *$\mathcal{P}$ is odd-M-pie-free;* (iii) *$\mathcal{P}$ is Mengerian.*

Proof. ((iii)$\Rightarrow$ (i)) Trivial. ((i)$\Rightarrow$ (ii)) Suppose that $\mathcal{P}$ is not odd-M-pie-free and let $\mathcal{P}' = \mathcal{P} \setminus A/B = \{P'_1, \dots, P'_k\}$ be an odd pie centered at c in $T/A \cup B$, where T supports $\mathcal{P}$. We can suppose that $|P'_i| = 2$, $i = 1, \dots, k$, by the following argument: let E' be the edge set of $T/A \cup B$; delete every $e \in E'$ which does not occur in any member of $\mathcal{P}'$ and contract every $e \notin E'(c)$. Thus $P'_i \cap \{e'_1, \dots, e'_k\} = \{e'_i, e'_{i+1}\}$, $i = 1, \dots, k$, where $e'_1, \dots, e'_k$ are incident to c in $T/A \cup B$. Hence $P'_i = \{e'_i, e'_{i+1}\}$, $i = 1, \dots, k$, that is, $\mathcal{P}'$ is isomorphic to the edge set of an odd polygon. Since the latter clutter is not Ideal, it follows that $\mathcal{P}$ is not Ideal. ((ii)$\Rightarrow$ (iii)). Suppose $\mathcal{P}$ is odd-M-pie-free but it is not Mengerian. By Theorem 1 one has $\nu_{2w}(\mathcal{P}) > 2\nu_w(\mathcal{P})$ for some $w \in \mathbb{Z}^E_+$. Let w be chosen so as to minimize $\sum_{e \in E} w(e)$ and let $E^* := \{e \in E \mid w(e) \geq 1\}$ be its support. Therefore, for $e \in E^*$, $\nu_{2(w-\chi_e)}(\mathcal{P}) = 2\nu_{w-\chi_e}(\mathcal{P})$, $\chi_e \in \mathbb{Z}^E_+$, being the incidence vector of edge e over E. Let $x \in \mathbb{Z}^{\mathcal{P}}_+$ be a $2w$-matching of size $\nu_{2w}(\mathcal{P})$ and let $\mathcal{M} = \{P \in \mathcal{P} \mid x(P) \geq 1\}$ be its support. The clutter $\mathcal{M}$ must contain some odd pie otherwise it would be Unimodular and we would have $\nu_{2w}(\mathcal{P}) = \nu_{2w}(\mathcal{M}) = 2\nu_w(\mathcal{M}) \leq 2\nu_w(\mathcal{P})$. Let $\{P_1, \dots P_k\} \subseteq \mathcal{M} \subseteq \mathcal{P}$ be any odd pie in T. Notice that $\cup_i P_i \subseteq E^*$. By Lemma 2, there are disjoint members L_0 and L_1 of $\mathcal{P}$ such that, for some $i = 1, \dots, k$, one has $L_j \subseteq P_i \Delta P_{i+1}, j = 0, 1$. Define $\bar{x}$ as follows:

$$\bar{x}(P) = \begin{cases} x(P) - 1 & \text{if } P \in \{P_i, P_{i+1}\} \\ x(P) + 1 & \text{if } P \in \{L_0, L_1\} \\ x(P) & \text{otherwise.} \end{cases}$$

By construction,

$$\sum_{P \ni e} \bar{x}(P) = \begin{cases} \sum_{P \ni e} x(P) - 1 & \text{if } e \in (P_i \Delta P_{i+1}) - (L_0 \cup L_1) \\ \sum_{P \ni e} x(P) - 2 & \text{if } e \in P_i \cap P_{i+1} \\ \sum_{P \ni e} x(P) & \text{otherwise.} \end{cases}$$

Since $P_i \cap P_{i+1}$ is nonempty (because it contains at least the edge $e_{i+1} \in E^*$ incident to the center of the pie), it follows that $\bar{x}$ is a $2(w - \chi_{e_{i+1}})$-matching of size

$$\sum_{P \in \mathcal{M} \cup \{L_0, L_1\}} \bar{x}(P) = \sum_{P \in \mathcal{M}} x(P),$$

contradicting the minimality of w. $\square$

Recall that a family $\mathcal{P}$ is *Helly* (equivalently, *has the Helly Property*) if pairwise intersecting members have a common element. A family $\mathcal{P}$ is *uniform* if all of its members have the same cardinality.

Corollary 1. *Let $\mathcal{P}$ be a Helly or uniform EPT family. Then $\mathcal{P}$ is Ideal if and only if it is Unimodular.*

Proof. The "if " part is trivial. Let us prove the "only if " part and let $\mathcal{P}$ be an Ideal EPT family. Without loss of generality $\mathcal{P}$ is a clutter. By Theorem 2, $\mathcal{P}$ is odd-M-pie-free. Lemma 1 implies that $\mathcal{P}$ is Unimodular if and only if it is odd-pie-free. Thus it suffices to prove that if $\mathcal{P}$ is odd-M-pie-free but it is not odd-pie-free then it is neither uniform nor Helly. Let $\{P_1, \dots, P_k\} \subseteq \mathcal{P}$ be an odd pie in the supporting tree T of $\mathcal{P}$. By Lemma 2 one has $L_0 \cup L_1 \subseteq P_i \Delta P_{i+1}$ for some $i = 1, \dots, k$, and some disjoint $L_0, L_1 \in \mathcal{P}$. $\mathcal{P}$ cannot be uniform otherwise we would get the contradiction $2r = |P_1| + |P_2| = |P_1 \Delta P_2| + 2|P_1 \cap P_2| > |L_0| + |L_1| = 2r$, r being the common cardinality of the members of $\mathcal{P}$. $\mathcal{P}$ cannot be Helly. Indeed, by the proof of the lemma, $\{L_0, P_i, P_{i+1}\}$ and $\{L_1, P_i, P_{i+1}\}$ are odd pies. Thus L_j, P_i and P_{i+1} pairwise meet but $L_j \cap P_i \cap P_{i+1} = \emptyset$, $j = 1, 2$. $\square$

Corollary 1 shows that EPT families are sharp generalizations of graphs when the integrality properties of matching and covering polyhedra are concerned. On the one hand the edge set of any graph $G = (V, E)$ is a uniform EPT family: take $T \cong K_{1,n}$ as supporting tree of E, n being the number of vertices of G; thus we have set a bijection between the vertices of G and the edges of T so that adjacent vertices of G correspond to adjacent edges of T. On the other hand it is well known that E is Ideal if and only if G is bipartite and hence Unimodular. Moreover, E is Totally Balanced if and only if G is a forest. Hence if E is Totally Balanced then it is Unimodular.

Summarizing, in this paper we have shown that odd pies are the only obstructions to the Total Dual Integrality of the defining system of (1). In this respect odd pies play the same role played by odd circuits in graphs: odd circuits in graphs are the only obstructions to the König Property.

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